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ELASTIC–PLASTIC ANALYSIS OF A FINITE SHEET WITH A COLD-WORKED HOLE

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Abstract—An exact elastic–plastic solution is obtained for the residual stress and strain field in a finite circular sheet having a cold-worked or interference fitted hole on the basis of J_2 deformation theory together with a modified Ramberg–Osgood law. Many factors influencing the residual stresses are then analysed. Further, comparison with finite element results and experimental data for rectangular sheets containing cold-worked holes is made. It is shown that the solution of a finite circular sheet can be used to predict the residual stresses on the minimum cross-section in a rectangular sheet with a cold-worked hole quickly and effectively, so long as the diameter of the circular sheet is equal to the width of the sheet.

1. INTRODUCTION

THERE EXIST numerous problems of circular holes subjected to uniform radial pressure, interference fitted load and cold-worked stresses in engineering. In particular, in the aeronautics industry, the development of local stress–strain methods and weight function techniques for stress intensity factors made the analysis of residual stresses one of the key problems in prolonging the life of structures by interference fit or cold-working techniques.

Many theoretical analyses have been done for the solution of a hole loaded by radial pressure or enlargement [1–5], but no solution has been found for a finite sheet with a strain hardening property with a hole. Thus, numerical methods and experimental techniques have to be used to obtain the stress field in a number of practical components with finite size, such as lugs with interference fitted bush, fasteners, and tubes bearing high pressure or fitted by a plastic enlargement into headplates. The numerical analyses and experimental measurements are time consuming. However, a fast and effective method for calculating the stress field is required in engineering.

Hsu and Forman [1] studied the applicability of J_2 deformation theory in the problem of an infinite sheet having a pressured hole in the light of Budiansky's criterion [6]. Before this, Mangasarian [5] carried out numerical analyses on the basis of J_2 deformation and incremental theories. He found that the J_2 deformation theory is completely satisfied in the case of normally loaded circular holes. Moreover, the numerical solutions of the two theories do not differ greatly even when the stress paths are far from being radial.

In the present paper, Budiansky's theory [2] is used to obtain exact solutions of a finite circular sheet having a hole subjected to internal pressure, interference fit load and cold-worked stresses, with the strain hardening behaviour and Baushinger's effect being considered. Then, the effectiveness of the solution to predict the stress and strain fields of some typical practical structures, such as cold-worked fasteners, is analysed.

2. ANALYSIS OF THE PROBLEM

Let the diameter of the hole be $2a$, the external diameter of the circular sheet and the width of the rectangular sheet be $2b$ (see Fig. 1). The modified Ramberg–Osgood model,

$$\varepsilon = \begin{cases} \dfrac{\sigma}{E} & \sigma < \sigma_Y \\ \dfrac{\sigma}{E}\left[\dfrac{\sigma}{\sigma_Y}\right]^{n-1} & \sigma > \sigma_Y, \end{cases} \quad (1)$$

is assumed, and the material is considered orthotropic but remains isotropic in the plane of the sheet. Here, the solution for the plane stress state is presented.

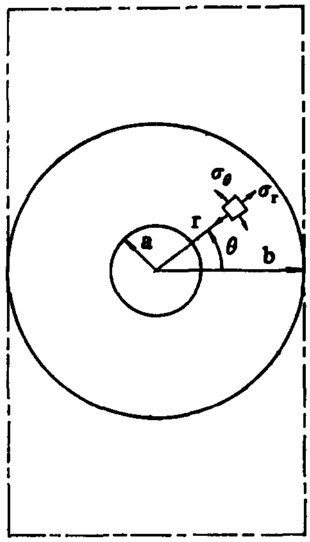

Fig. 1. Schematic diagram of the problem.

2.1. *Basic equations*

The shearing stress and strain, $\tau_{r\theta}$ and $\gamma_{r\theta}$, are zero by symmetry, and the radial and tangential stresses σ_r and σ_θ must satisfy the equilibrium equation,

$$\frac{d\sigma_r}{dr}+\frac{\sigma_r-\sigma_\theta}{r}=0, \tag{2}$$

and the corresponding strains ε_r and ε_θ, given by

$$\varepsilon_r=\frac{du_r}{dr},\quad \varepsilon_\theta=\frac{u_r}{r}, \tag{3}$$

in terms of the radial displacement u_r must satisfy the compatibility equation

$$\frac{d\varepsilon_\theta}{dr}+\frac{\varepsilon_\theta-\varepsilon_r}{r}=0. \tag{4}$$

Decomposing the total strain as

$$\varepsilon_r=\varepsilon_r^e+\varepsilon_r^p;\quad \varepsilon_\theta=\varepsilon_\theta^e+\varepsilon_\theta^p, \tag{5}$$

the elastic strains are then related to the stresses by

$$\varepsilon_r^e=(\sigma_r-\nu'\sigma_\theta)/E;\quad \varepsilon_\theta^e=(\sigma_\theta-\nu'\sigma_r)/E, \tag{6}$$

with ν' the Poisson ratio of the elastic strains, and the plastic strains are

$$\varepsilon_r^p=\left(\frac{1}{Es}-\frac{1}{E}\right)\left[\sigma_r-\frac{R}{1+R}\sigma_\theta\right]$$
$$\varepsilon_\theta^p=\left(\frac{1}{Es}-\frac{1}{E}\right)\left[\sigma_\theta-\frac{R}{1+R}\sigma_r\right]. \tag{6a}$$

Here R is the ratio of the transverse plastic strain in the plane of the sheet to the plastic strain through the thickness, and Es the secant modulus on the uniaxial stress–strain curve at the effective stress σ,

$$\sigma=\left[\sigma_r^2+\sigma_\theta^2-\frac{2R}{1+R}\sigma_r\sigma_\theta\right]^{1/2}, \tag{7}$$

and from eq. (1) we get

$$\frac{1}{Es}=\begin{cases}1/E & \text{for } 0<\sigma<\sigma_Y\\ (\sigma/\sigma_Y)^{n-1}/E & \text{for } \sigma>\sigma_Y.\end{cases} \tag{8}$$

As v' has no effect on the solution of stress, $v' = R/1 + R$ is chosen, together with eqs (5)–(6a), to give

$$\varepsilon_r = \left[\sigma_r - \frac{R}{1+R}\sigma_\theta\right]\bigg/ Es$$

$$\varepsilon_\theta = \left[\sigma_\theta - \frac{R}{1+R}\sigma_r\right]\bigg/ Es. \tag{9}$$

2.2. *Stress solution*

Firstly, we consider the internal pressure of the hole. The boundary conditions for this case are

$$\sigma_r(a) = -q_a; \quad \sigma_r(b) = 0. \tag{10}$$

The elastic solutions are

$$\sigma_r = -\frac{b^2/r^2 - 1}{b^2/a^2 - 1} q_a$$

$$\sigma_\theta = \frac{b^2/r^2 + 1}{b^2/a^2 - 1} q_a. \tag{11}$$

When the sheet is subjected to elastic–plastic deformation, the solutions in the elastic region ($r_p \leqslant r \leqslant b$) can be given as

$$\sigma_r = -\frac{b^2/r^2 - 1}{b^2/a^2 - 1} q_p$$

$$\sigma_\theta = \frac{b^2/r^2 + 1}{b^2/a^2 - 1} q_p. \tag{12}$$

Here, q_p is the pressure of the plastic domain on the boundary of the elastic domain, $r = r_p$.

At $r = r_p$, equilibrium of stress and the continuity of u_r requires continuity of σ_r and ε_θ, and further, of σ_θ according to (2) and σ according to (7); thus, substituting (12) into (7) with $\sigma = \sigma_r$ gives

$$q_p = \sigma_Y\left(\frac{b^2}{a^2} - 1\right)\left[2\frac{b^4}{r_p^4} + 2 + \frac{2R}{1+R}\left(\frac{b^4}{r_p^4} - 1\right)\right]^{-1/2}. \tag{13}$$

Then, substituting (13) into (12) yields the stress solutions in the elastic domain:

$$\sigma_r = -\sigma_Y\left(\frac{b^2}{r^2} - 1\right)\left[2\frac{b^4}{r_p^4} + 2 + \frac{2R}{1+R}\left(\frac{b^4}{r_p^4} - 1\right)\right]^{-1/2}$$

$$\sigma_\theta = \sigma_Y\left(\frac{b^2}{r^2} + 1\right)\left[2\frac{b^4}{r_p^4} + 2 + \frac{2R}{1+R}\left(\frac{b^4}{r_p^4} - 1\right)\right]^{-1/2}. \tag{14}$$

In the plastic domain ($a \leqslant r \leqslant r_p$), Budiansky's solution with a parameter α is introduced:

$$\sigma_r = \sigma\sqrt{\left(\frac{1+R}{2}\right)}\left[\cos\alpha - \frac{1}{\sqrt{(1+2R)}}\sin\alpha\right]$$

$$\sigma_\theta = \sigma\sqrt{\left(\frac{1+R}{2}\right)}\left[\cos\alpha + \frac{1}{\sqrt{(1+2R)}}\sin\alpha\right]. \tag{15}$$

Combining (14) and (15) yields the equations about α at $r = r_p$, α_p, with $\sigma = \sigma_Y$:

$$\sin\alpha_p = \sqrt{(1+2R)}\left(\frac{b}{r_p}\right)^2\left[(1+2R)\left(\frac{b}{r_p}\right)^4 + 1\right]^{-1/2}$$

$$\cos\alpha_p = \left[(1+2R)\left(\frac{b}{r_p}\right)^4 + 1\right]^{-1/2}, \tag{16}$$

where $0 < \alpha_p < \pi/2$.

The compatibility equation can be expressed in stresses from (4) and (9):

$$Esd\left\{\frac{1}{Es}\left(\sigma_\theta-\frac{r}{1+R}\sigma_r\right)\right\}+d(\sigma_\theta+\sigma_r)=0. \tag{17}$$

Substituting (15) into (17) and considering (7) gives

$$\int_{\sigma_Y}^{\sigma}\frac{1}{\sigma}\,d\sigma=\int_{\alpha_p}^{\alpha}\frac{2(1+R)\sin\alpha}{(n+1+2R)\cos\alpha+(n-1)\sqrt{(1+2R)}\sin\alpha}\,d\alpha.$$

Carrying out the integration yields

$$\frac{\sigma}{\sigma_Y}=\left[\frac{a_1\sin\alpha_p+a_2\cos\alpha_p}{a_1\sin\alpha+a_2\cos\alpha}\right]^{\mu}\exp\left[\frac{2a_1(1+R)(\alpha-\alpha_p)}{a_1^2+a_2^2}\right], \tag{18}$$

where $a_1=(n-1)\sqrt{(1+2R)}$, $a_2=(n+1+2R)$ and $\mu=2a_2(1+R)/(a_1^2+a_2^2)$.

Let $\alpha=\alpha_a$ at $r=a$. Then from (10), (15) and (18) we get

$$q_a=\sigma_Y\sqrt{\left(\frac{1+R}{2}\right)}\left[\frac{\sin\alpha_a}{\sqrt{(1+2R)}}-\cos\alpha_a\right]\left[\frac{a_1\sin\alpha_p+a_2\cos\alpha_p}{a_1\sin\alpha_a+a_2\cos\alpha_a}\right]^{\mu}\exp\left[\frac{2a_1(1+R)(\alpha_a-\alpha_p)}{a_1^2+a_2^2}\right]. \tag{19}$$

Combining (2) and (15), the resulting form, with the help of (18), gives a first-order differential equation in r and α. Integrating it with $\alpha=\alpha_a$ at $r=a$ yields

$$\frac{r}{a}=\sqrt{\left(\frac{\sin\alpha_a}{\sin\alpha}\right)}\left[\frac{a_1\sin\alpha+a_2\cos\alpha}{a_1\sin\alpha_a+a_2\cos\alpha_a}\right]^{\gamma}\times\exp\left[\frac{(n^2-1)\sqrt{(1+2R)}(\alpha_a-\alpha)}{2(n^2+1+2R)}\right], \tag{20}$$

where $\gamma=[n(1+R)]/[n^2+1+2R]$.

At $r=r_p$, $\alpha=\alpha_p$ and from (20) we get

$$\frac{r_p}{a}=\sqrt{\left(\frac{\sin\alpha_a}{\sin\alpha_p}\right)}\left[\frac{a_1\sin\alpha_p+a_2\cos\alpha_p}{a_1\sin\alpha_a+a_2\cos\alpha_a}\right]^{\gamma}\times\exp\left[\frac{(n^2-1)\sqrt{(1+2R)}(\alpha_a-\alpha_p)}{2(n^2+1+2R)}\right]. \tag{21}$$

Combining (21), (19) and (16) forms a set of transcendental equations from which α_a, α_p and r_p can be solved, with $\alpha_p<\alpha_a<\alpha_m$, where α_m is given by (22). After α_a, α_p and r_p are determined, the stresses at any point in the plastic domain can be found from (20), (18) and (15), and the stresses in the elastic range can be obtained from (14).

When $b\to\infty$ and $q_a\to\infty$, $r_p\to\infty$ and the maximum of α can be obtained from (21):

$$\alpha_m=\tan^{-1}\left[\frac{n+1+2R}{(1-n)\sqrt{(1+2R)}}\right]. \tag{22}$$

2.3. *Strain and displacement solution*

In the elastic domain, substituting (14) into (6) gives

$$\varepsilon_r^e=\frac{\sigma_Y}{E}\left[1-v'-(1+v')\frac{b^2}{r^2}\right]\left[2\frac{b^4}{r_p^4}+2+\frac{2R}{1+R}\left(\frac{b^4}{r_p^4}-1\right)\right]^{-1/2}$$

$$\varepsilon_\theta^e=\frac{\sigma_Y}{E}\left[1-v'+(1+v')\frac{b^2}{r^2}\right]\left[2\frac{b^4}{r_p^4}+2+\frac{2R}{1+R}\left(\frac{b^4}{r_p^4}-1\right)\right]^{-1/2}, \tag{23}$$

and from (3)

$$u_r=r\varepsilon_\theta^e.$$

In the plastic domain, substituting (15) into (5) and (6) yields

$$\varepsilon_r=\frac{\sigma}{Es}\sqrt{\left(\frac{1+R}{2}\right)}\left[(1-v)\cos\alpha-\frac{1+v}{\sqrt{(1+2R)}}\sin\alpha\right]$$

$$\varepsilon_\theta=\frac{\sigma}{Es}\sqrt{\left(\frac{1+R}{2}\right)}\left[(1-v)\cos\alpha+\frac{1+v}{\sqrt{(1+2R)}}\sin\alpha\right] \tag{24}$$

and

$$u_r=r\varepsilon_\theta.$$

At the edge of the hole ($r = a$),

$$u_r(a) = a\frac{\sigma}{Es}\sqrt{\left(\frac{1+R}{2}\right)}\left[(1-\nu)\cos\alpha + \frac{1+\nu}{\sqrt{(1+2R)}}\sin\alpha\right] \tag{25}$$

where ν is a function of the ratio of plastic strain to elastic strain, and $\nu' \leqslant \nu \leqslant \frac{1}{2}$.

2.4. *Discussion*

2.4.1. *Solution for a circular sheet with an expanded hole.* When the radius of the hole is expanded from r to $r + U_a$, substituting (18) into (25) and taking $\alpha = \alpha_a$ gives

$$U_a = a\frac{\sigma_Y}{E}\sqrt{\left(\frac{1+R}{2}\right)}\left[(1-\nu)\cos\alpha_a + \frac{1+\nu}{\sqrt{(1+2R)}}\sin\alpha_a\right] \times \left\{\left[\frac{a_1\sin\alpha_p + a_2\cos\alpha_p}{a_1\sin\alpha_a + a_2\cos\alpha_a}\right]^{\mu} \times \exp\left[\frac{2a_1(1+R)(\alpha_a-\alpha_p)}{a_1^2+a_2^2}\right]\right\}^{n}. \tag{26}$$

Combining (16), (21) and (26) we solve α_a, α_p and r_p.

2.4.2. *Solution for the hole interference fitted with an elastic bolt.* Let the elastic modulus of the bolt be E_0, Poisson's ratio be ν_0 and its diameter be $b_0 = a + U_a$. Then the general solution of the bolt can be expressed as

$$\sigma_r = 2C \tag{27}$$

$$u_r = 2C\frac{1-\nu_0}{E_0}r, \tag{28}$$

where C is an unknown constant. Considering the continuity of σ_r on the contacted face between the hole and bolt, from (27) and (15) we get

$$2C = \sigma\sqrt{\left(\frac{1+R}{2}\right)}\left[\cos\alpha_a - \frac{1}{\sqrt{(1+2R)}}\sin\alpha_a\right]. \tag{29}$$

Then, considering the compatibility condition of displacement on the contacted face,

$$u_r(a) - u_r(b_0) = U_a, \tag{30}$$

we find by substituting (29), (28) and (26) into (30) that

$$\frac{u_a E}{\sigma} = a\left(\frac{\sigma}{\sigma_Y}\right)^{n-1}\left[(1-\nu)\cos\alpha_a + \frac{1+\nu}{\sqrt{(1+2R)}}\sin\alpha_a\right] - (1-\nu_0)\frac{b_0 E}{aE_0}\left(\cos\alpha_a - \frac{1}{\sqrt{(1+2R)}}\sin\alpha_a\right). \tag{31}$$

The corresponding α_a, α_p and r_p can then be solved by combining (16), (31) and (21).

2.4.3. *Residual stresses in a circular sheet with a cold-worked hole.* Residual stresses and the permanent residual enlargement for a circular sheet having a cold-worked hole can be solved in a similar way as in ref. [1]; however, the reverse yield in the material during unloading is considered here.

For points not yielding during the loading phase, the reverse yield stress is determined by

$$\sigma_Y^- = \sigma^+ + \sigma_Y, \tag{32}$$

and for points bearing plastic deformation as loading,

$$\sigma_Y^- = 2(1-H)\sigma_Y + 2H\sigma^+, \tag{33}$$

where σ^+ stands for the maximum effective stress achieved during the loading process. H is the parameter for Baushinger's effect of the material and $0 \leqslant H \leqslant 1$.

3. ANALYSIS OF THE RESULTS

An iterative process was used on the transcendental equation set to solve α_a, α_p and r_p. The procedure converged very rapidly.

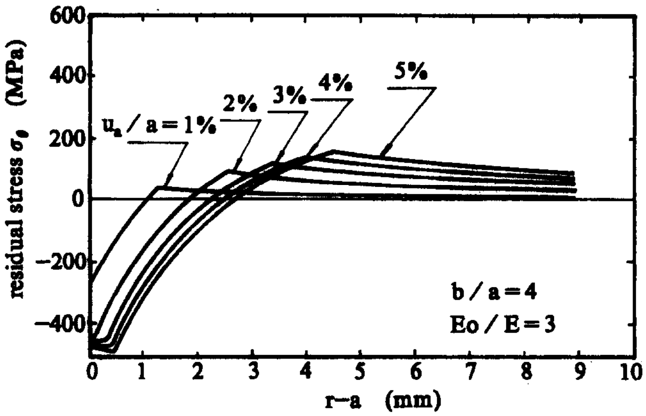

Fig. 2. The effect of interference on the distribution of residual stress in LY12CZ aluminium plate.

Figure 2 shows the effects of difference interference on the residual stress field. It can be seen that the plastic range expands and the maximum tangential stresses and residual stresses increase as the relative interference U_a/a increases. What is more, the residual stresses increase quickly with U_a/a when the interference is low, but very slowly at high interference.

It can be seen from Fig. 3 that the size ratio of the sheet, b/a, has obvious effects on the residual stress field, but when b/a becomes large its influence becomes weak in the range of $r < r_p$.

Effects of the ratio E_0/E on stress and the residual stress field are similar to that of interference (see Fig. 4) and become weaker as the interference becomes larger.

The hardening parameter n exerts a strong influence on the residual stress field when n is not very large, and the influence diminishes as n increases, as shown in Fig. 5.

4. DISCUSSION

In engineering practice, interference fitted and cold-worked fasteners are widely used to prolong the life of structures, and a rectangle is the most typical fastener shape. Here the suitability of the analytical solution for a circular sheet to a rectangular fastener will be discussed.

Figure 6 gives Crows' finite element (FE) results [4] for a 7075-T6 aluminium alloy rectangular sheet with an interference fitted hole, together with the present solutions for a circular sheet with the same b. It can be seen that a good agreement exists between them, especially for $r < r_p$.

A comparison of the present solution of residual stresses with the FE results for a high strength steel circular sheet having a cold-worked hole is shown in Fig. 7. The two solutions coincide very well. The corresponding FE solutions for a rectangular sheet with the same b are also presented in Fig. 7. Though the plastic range is very large, good agreement is still found for the residual stress component σ_θ.

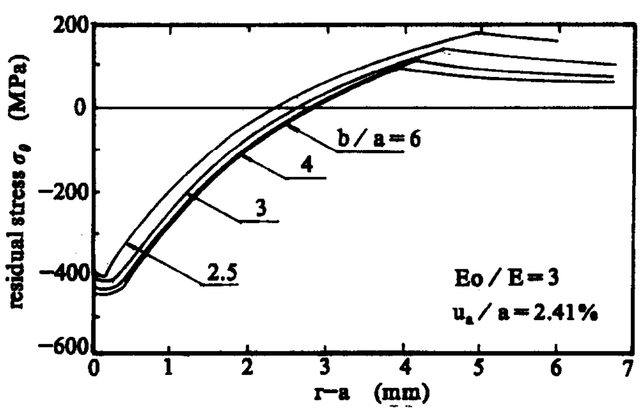

Fig. 3. The effect of b/a on the distribution of residual stress in LY12CZ aluminium plate.

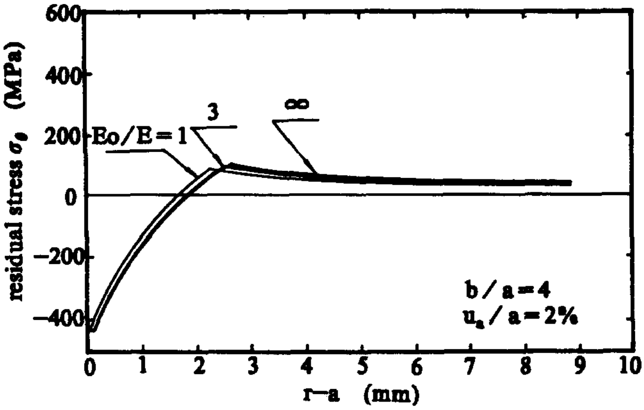

Fig. 4. The effect of E_0/E on the distribution of residual stress in LY12CZ aluminium plate.

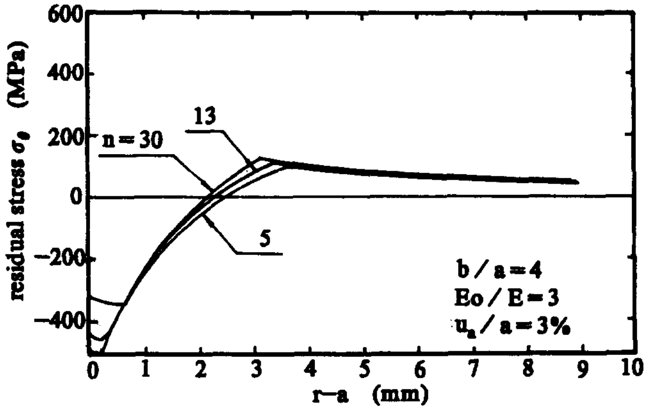

Fig. 5. The effect of n on the distribution of residual stress.

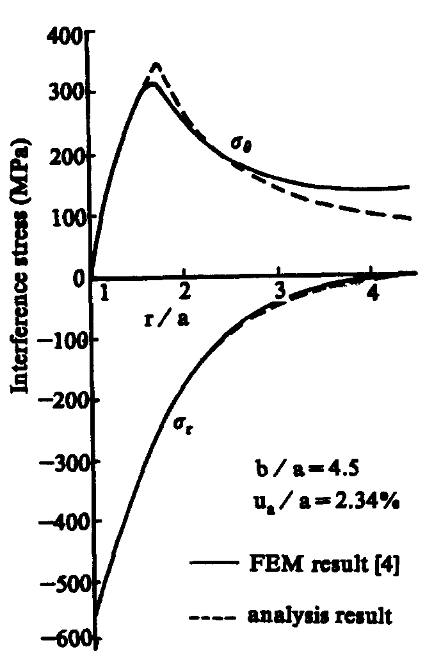

Fig. 6. Interference stresses on the section of $\theta = 0°$ in 7075 aluminium plate.

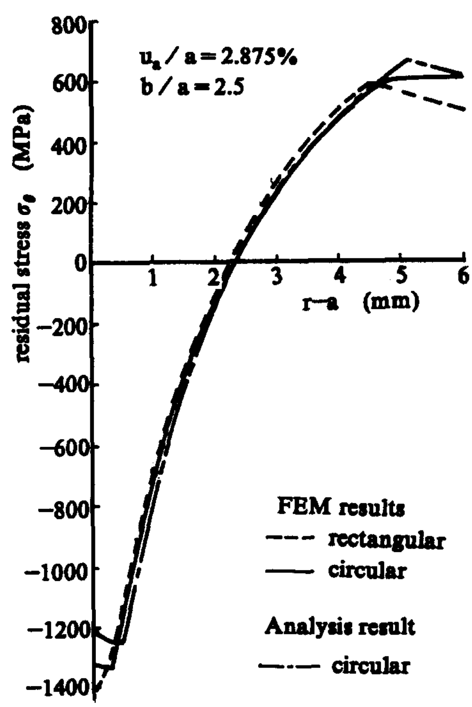

Fig. 7. The distribution of residual stress in a high strength steel.

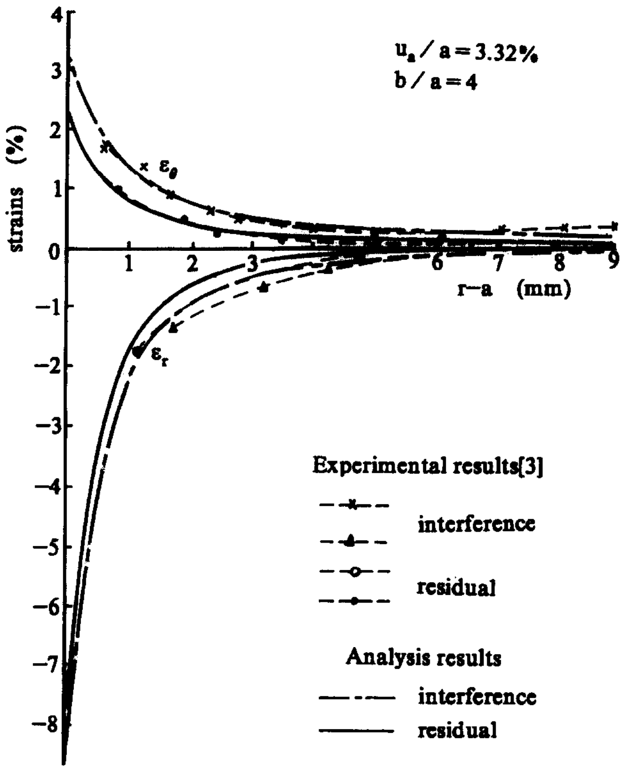

Fig. 8. The distribution of residual strains in LY12CZ aluminium plate.

In Fig. 8, a comparison of the present results with experimental data of interference strains and residual strains in a rectangular sheet is presented. A good agreement is obtained once more. The present solution gives $r_p = 6.64$ mm, and the experiment [3] yields r_p between 6.5 and 6.6 mm; the error is less than 2%.

In conclusion, it can be said that the present solution for a circular sheet can be used to predict the residual stress field in a rectangular fastener with an interference fitted or cold-worked hole quickly and effectively, provided they have the same value of b.

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