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ELASTOPLASTIC THREE DIMENSIONAL CRACK BORDER FIELD—I. SINGULAR STRUCTURE OF THE FIELD

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Abstract—The structure of stress and strain fields at the border of three dimensional cracks in a tension field is investigated for elastoplastic materials treated by a deformation theory. The investigation is based upon the physics of the problem and is conducted with mathematical rigour. It is found that the character of singular stresses is as follows:

$$\sigma_{ij} = r^{f(z)-2}\tilde{\sigma}_{ij}(\theta, Tz) \quad (i,j = x,y),$$

where $f(z)$ is a function of triaxial stress constraint Tz. The transverse shear stresses σ_{yz} and σ_{xz} are of the order of unity. The corresponding in-plane strains ε_{ij} $(i,j = x,y)$ have singularity of order $n(f(z)-2)$, while ε_{yz} and ε_{xz} are of the order of unity. ε_{zz} has the same order as in-plane strains at corner points but may be much weaker in the interior of the crack border. Further, it is argued that the problem can be simplified to a quasi-planar problem with the triaxial stress constraint Tz being considered.

When the solution is degenerated into a plane problem by enforcing the confinement, the exact solution for a plane strain crack is obtained and some interesting phenomena are discussed in detail.

1. INTRODUCTION

FRACTURE in structural components of practical dimensions made of low-to-intermediate strength metals is often accompanied by extensive plastic deformation prior to crack extension. Hence, one is compelled to seek elastoplastic solutions to crack problems to treat such situations.

For two dimensional (2D) configurations, the asymptotic crack tip solution for power hardening materials was developed by Hutchinson [1] and Rice and Rosengren [2], and has become well known as the HRR solution. Since then, an enormous amount of numerical and experimental research has been conducted to justify the suitability of HRR solution as well as the dominance of Rice's path independent J-integral in real specimens [3–6]. It is shown that the HRR solution coincides very well with finite element results and J dominates the crack tip field effectively in a plane stress state, while in a plane strain state the J dominant region is much smaller and changes with geometrical and loading configurations [3–7]. For three dimensional (3D) cracks, recent research shows a tendency toward loss of the J dominance [8, 9].

Much effort has been made to obtain a better description of the plane strain crack tip field [10–14]. One important theory about this problem is that the region of dominance of the leading term in the expansion of the solution may not be sufficiently large and the second term should be introduced. Li and Wang [10], Betegon and Hancock [11], Sharma and Aravas [12], O'Dowd and Shih [13, 14], etc. have made important contributions in this field, and O'Dowd and Shih have developed this idea into a J–Q annulus family in their work, i.e.

$$\frac{\sigma_{ij}(r,\theta)}{\sigma_0} = \left(\frac{J}{\alpha\varepsilon_0\sigma_0 I_n}\right)^{1/(n-1)}\tilde{\sigma}_{ij}(\theta) + Q\left(\frac{r}{J/\sigma_0}\right)^{\lambda}\hat{\sigma}_{ij}(\theta),$$

where Q is a function of the stress triaxiality achieved ahead of the plane strain cracks.

Under a general triaxial stress constraint, the problem of elastoplastic cracks becomes very complicated. Although some valuable theoretical attempts and a large amount of numerical research have been made for 3D situations [8, 9, 15–18], it appears that no substantial breakthrough has been made regarding the singular character of 3D elastoplastic crack border fields.

It has been shown that under a triaxial stress constraint, the local distribution of stresses and strains near the crack border is closely connected to the external loading configuration and cannot be characterized by a single parameter [7–9]. Therefore, the asymptotic analysis technique used in 2D problems may not be efficient. On the other hand, it seems to be impossible to find the global and analytic solutions for such a complicated problem.

This situation led the present author to believe that a combination of strict mathematical analysis and a good understanding of the physical substance of 3D cracks is necessary for the problem. In this paper, the essential physical fundamentals of the problem as well as the character of a triaxial stress constraint near a 3D crack border are first discussed. Then two general hypotheses are proposed based on the triaxial stress constraint and the support of available numerical analyses. Following these, the structures of stresses and strains in 3D mode I crack border fields are determined and the problem can be simplified to a quasi-planar problem with the constraint of stress triaxiality being considered. This makes it possible to solve the problem analytically.

2. THE PHYSICAL SUBSTANCE OF THE PROBLEM AND THE TRIAXIAL STRESS CONSTRAINT

2.1. *Definitions*

For the convenience of study, the coordinate system shown in Fig. 1 is adopted, where P is a point of the curve defining the crack front, and x, y and z are the normal, binormal and tangent components at P, respectively. The plane defined by the normal and tangent, or plane x–z, will be assumed to coincide with the tangent to the free surfaces of the crack at P. The plane defined by x and y is the normal plane, and r and θ are polar coordinates in this plane.

For a through thickness straight crack, which is a typical case of 3D cracks, the point P is located at the center of the specimen for convenience, as shown in Fig. 1b.

Taking the thin sheet element lying in the x–y plane as the object of study, the loading configuration of the element for mode I cracks is as illustrated in Fig. 1c.

Obviously, that the stress field in the sheet element differs from that in the plane stress state is caused by nothing but the presence of a stress constraint in the z direction, σ_{zz}. Therefore, σ_{zz} is the direct reason to raise the 3D crack problem. Thus it is reasonable to define the triaxial stress constraint Tz as

$$Tz = \frac{\sigma_{33}}{\sigma_{11} + \sigma_{22}}, \tag{1}$$

where the subscripts 1, 2 and 3 stand for x, y and z or r, θ and z, respectively.

2.2. *Physical substance of the problem*

In the plane stress state, $Tz = 0$ and the region of dominance of HRR solution and J-integral is always sufficiently large. In the plane strain state, Tz may decrease from 0.5 at the tip to ν as the distance from the tip, r, increases. At the same time, the difference between the HRR solution (where $Tz = 0.5$) and the field in real specimens increases and J dominance decreases rapidly as r increases [8, 9, 13]. In a general 3D state, Tz ranges from 0 to 0.5 at different points, and the single-parameter-dominance is no longer suitable [8, 9, 15]. These limitations of the 2D solution and J-integral have been proved by a large amount of fact (as mentioned above), but what is the essential reason?

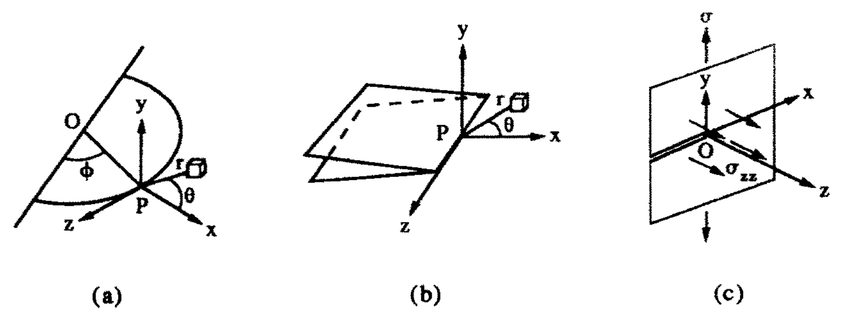

Fig. 1. Definition of coordinates and the sheet element in the normal plane x–y. (a) Coordinate for a curved crack; (b) coordinate for a straight through thickness crack; (c) loading configuration of the sheet element.

Phenomenally, the crack border field in a plane strain and 3D state is affected by crack tip blunting, large plastic deformation, load levels and the geometry of specimens. So, it is easy to fall into the conclusion that the limitation of deformation theory and the assumption of an ideal sharp crack are the essential reasons for the defects of the HRR solution. However, consider the fact that crack tip blunting and plastic deformation are severe in plane stress as well, but their influence and the effects of external loading configuration upon the crack tip field (for $r > 2J/\sigma_0$) are very weak, and J dominates the field in a large region. With this fact in mind one has to admit that neither crack tip blunting nor the limitation of deformation theory is the essential reason for the mentioned limitations of HRR solution when $Tz \neq 0$.

Let us consider the sheet element shown in Fig. 1c. It is obvious that the only difference between plane stress, plane strain and 3D cracks is the variation of Tz. In the plane stress HRR solution, $Tz \equiv 0$ and this fits the real condition. But in the plane strain HRR solution, Tz takes the upper limit of the triaxial stress constraint ahead of a real specimen, so it leads to a limit distribution of crack border fields (see [6, 13]).

Consequently, all of the problems raised in the description of 3D crack border fields are due to the presence of the triaxial constraint Tz and the use of Tz as a clue may be advantageous to the study of the problem.

2.3. *Characteristics of the triaxial stress constraint near 3D crack border*

Many elastoplastic numerical analyses for 3D cracked bodies have already been conducted for various purposes [8, 9, 15, 17, 18] and have been reviewed by Guo [19]. From these numerical results valuable information can be obtained about the distribution of the triaxial stress constraint near crack borders.

(1) Generally, Tz is a function of r, θ and z

$$Tz = Tz(r, \theta, z), \tag{2}$$

although the variation of Tz with θ is somewhat slight.

(2) When $r \to 0$, $Tz = T(z)$ and for elastic cracks $0 \leqslant T(z) \leqslant \nu$ and for an elastoplastic crack $0 \leqslant T(z) \leqslant 0.5$.

(3) When $r \to 0$, the differentiation of Tz with respect to r, $\partial Tz/\partial r$, may be great and should not be neglected.

(4) In the interior of the crack Tz is higher and its change in z direction is slight, while when the free surface is approached, Tz decreases rapidly and at the free surface $Tz = 0$.

(5) Tz is symmetrical about the plane of symmetry of geometry and load.

Furthermore, Tz is dependent upon load conditions and specimen geometries or is undetermined by the asymptotic field, so that when $Tz \neq 0$, the asymptotic field will no longer be a single-parameter characterized field and a two-parameter dominated field must be established, and Tz will be one of the proposed parameters.

3. SINGULAR STRUCTURE OF 3D CRACK BORDER FIELDS

3.1. *Basic equations*

For a 3D isotropic continuum without body force, the equilibrium equations are written as

$$\sigma_{ij,j} = 0, \tag{3}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress tensor.

The relation between the infinitesimal strain tensor $\boldsymbol{\varepsilon}$ and the displacement vector $\mathbf{u}$ is

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}). \tag{4}$$

The corresponding compatibility equations are given as

$$e_{mlk} e_{nij} \varepsilon_{ki,jl} = 0, \tag{5}$$

where

$$e_{ijk} = \tfrac{1}{2}(i-j)(j-k)(k-i).$$

When Maxwell stress functions Φ_{ii}, $(i = 1, 2, 3)$, are introduced, the stress tensor satisfying the equilibrium equations (3) can be expressed as

$$\sigma_{mn} = e_{mkj}e_{nij}\Phi_{ji,kl}. \tag{6}$$

The constitutive behavior of the homogeneous isotropic elastoplastic continuum is described by the J_2-deformation theory for a Ramberg–Osgood uniaxial stress–strain behavior, namely

$$\varepsilon_{ij} = \frac{1+\nu}{E}S_{ij} + \frac{1-2\nu}{3E}\sigma_{kk}\delta_{ij} + \tfrac{3}{2}\alpha\varepsilon_0\left(\frac{\sigma_e}{\sigma_0}\right)^{n-1}\frac{S_{ij}}{\sigma_0}, \tag{7}$$

where S_{ij} is the stress deviator, δ_{ij} is the Kronecker delta, ν is Poisson's ratio, E is Young's modulus, α is a material constant, n is the hardening exponent $(1 \leqslant n \leqslant \infty)$, σ_0 is the yield stress, $\varepsilon_0 = \sigma_0/E$ and σ_e is the von Mises equivalent stress, defined as

$$\sigma_e^2 = \tfrac{3}{2}S_{ij}S_{ij}. \tag{8}$$

3.2. *Basic hypotheses*

Hypothesis 1. As analysed in Section 2, the out-of-plane stress σ_{zz} is related to the in-plane stresses by

$$\sigma_{zz}(r,\theta,z) = Tz(\sigma_{xx} + \sigma_{yy}) \tag{9}$$

$$Tz = Tz(r,\theta,z), \tag{10}$$

where $Tz \to T(z)$ when $r \to 0$ and $0 \leqslant T(z) \leqslant 0.5$.

Hypothesis 2. In HRR solution, it has been shown that when $Tz = 0$ or $\frac{1}{2}$, the order of singularity of stresses can be determined through Rice's path independent J-integral as there is no energy provided in the z direction, and the angular distribution of stresses is independent of r. For general 3D problems, Tz is between 0 and $\frac{1}{2}$, and the energy in the third direction will not disappear. In such a case, Rice's J-integral will no longer be path independent. Therefore, the order of singularity of stresses may differ from that in HRR. Without loss of generality, the singular exponents of stresses σ_{ij} should be assumed as functions of z, or $f_{ij}(z)$.

On the other hand, it has been proved by Guo [19] that for arbitrary r_1, r_2 and θ_1, θ_2 near the crack tip, the parameter

$$\delta_r = \frac{\dfrac{\sigma_{ij}(r_1,\theta_1)}{\sigma_{ij}(r_1,\theta_2)} - \dfrac{\sigma_{ij}(r_2,\theta_1)}{\sigma_{ij}(r_2,\theta_2)}}{\max\left\{\dfrac{\sigma_{ij}(r_1,\theta_1)}{\sigma_{ij}(r_1,\theta_2)}, \dfrac{\sigma_{ij}(r_2,\theta_1)}{\sigma_{ij}(r_2,\theta_2)}\right\}}$$

evaluated by finite element method reaches 45% on planes of z = const. This means that, for given z

$$\frac{\sigma_{ij}(r_1,\theta_1)}{\sigma_{ij}(r_1,\theta_2)} \neq \frac{\sigma_{ij}(r_2,\theta_1)}{\sigma_{ij}(r_2,\theta_2)}. \tag{11}$$

Then it is easy to prove that σ_{ij} cannot be expressed in a separable variable form. σ_{ij} are still related to r by the effect of Tz.

Actually, the fact that the angular distributions of stresses of plane stress, plane strain and 3D cracks are different to each other shows that $\tilde{\sigma}_{ij}$ are functions of triaxial stress constraint Tz.

As a result, hypothesis 2 can be made as

$$\sigma_{ij}(r,\theta,z) = r^{f_{ij}(z)}\tilde{\sigma}_{ij}(\theta, Tz), \tag{12}$$

where $f_{ij}(z)$ are functions of $T(z)$, and so are dimensionless.

3.3. *Singularities of stresses*

Let

$$x = r\cos\theta; \quad y = r\sin\theta; \quad z = z. \tag{13}$$

Then, combining eq. (6) and hypothesis 2 gives

$$\sigma_{xy} = -\frac{\partial^2 \Phi_3}{\partial x\, \partial y} = r^{f_{xy}(z)}\tilde{\sigma}_{xy}(\theta, Tz) \tag{14}$$

$$\sigma_{yz} = -\frac{\partial^2 \Phi_1}{\partial y\, \partial z} = r^{f_{yz}(z)}\tilde{\sigma}_{yz}(\theta, Tz) \tag{15}$$

$$\sigma_{xz} = -\frac{\partial^2 \Phi_2}{\partial x\, \partial z} = r^{f_{xz}(z)}\tilde{\sigma}_{xz}(\theta, Tz), \tag{16}$$

where $\Phi_i = \Phi_{ii}$.

Considering (13), eq. (14) can be expanded as

$$r^{f_{xy}(z)}\sigma_{xy}(\theta, Tz) = \frac{\partial^2 \Phi_3}{\partial r^2}\cos\theta \sin\theta + 2\frac{\partial^2 \Phi_3}{\partial \theta\, \partial r}\frac{\cos 2\theta}{r} - \frac{\partial \Phi_3}{\partial r}\frac{\sin\theta\cos\theta}{r} - 2\frac{\partial \Phi_3}{\partial \theta}\frac{\cos 2\theta}{r^2} - \frac{\partial^2 \Phi_3}{\partial \theta^2}\frac{\sin\theta\cos\theta}{r^2}. \tag{17}$$

Comparing both sides of (17), it can be seen that the dominant term of Φ_3 has the form of

$$\Phi_3 = r^{f_{xy}(z)+2}\tilde{\Phi}_3(\theta, Tz). \tag{18}$$

Similarly, from eq. (15) it can be obtained that

$$r^{f_{yz}(z)}\sigma_{yz}(\theta, Tz) = -\frac{\partial^2 \Phi_1}{\partial z\, \partial r}\sin\theta + \frac{1}{r}\frac{\partial^2 \Phi_1}{\partial \theta\, \partial z}\cos\theta. \tag{19}$$

For the convenience of analysis, let

$$\Phi_1 = r^{f_1(z)}\tilde{\Phi}_1(\theta, Tz). \tag{20}$$

Then

$$\frac{\partial^2 \Phi_1}{\partial z\, \partial r} = f_1(z) r^{f_1(z)-1}\left(\frac{\partial \tilde{\Phi}_1}{\partial z} + \tilde{\Phi}_1 \frac{\partial f_1(z)}{\partial z}\ln r\right) + r^{f_1(z)}\frac{\partial^2 \tilde{\Phi}_1}{\partial z\, \partial r} + r^{f_1(z)}\frac{\partial \tilde{\Phi}_1}{\partial r}\frac{\partial f_1(z)}{\partial z}\ln r \tag{21}$$

$$\frac{\partial^2 \Phi_1}{\partial z\, \partial \theta} = r^{f_1(z)}\frac{\partial^2 \tilde{\Phi}_1}{\partial z\, \partial \theta} + r^{f_1(z)}\frac{\partial \tilde{\Phi}_1}{\partial \theta}\frac{\partial f_1(z)}{\partial z}\ln r. \tag{22}$$

Substituting (21) and (22) into (19) gives

$$r^{f_{yz}(z)}\sigma_{yz}(\theta, Tz) = r^{f_1(z)}\frac{\partial f_1(z)}{\partial z}\ln r\tilde{\Phi}_1^{*}(\theta, Tz) + r^{f_1(z)}\frac{\partial^2 \tilde{\Phi}_1}{\partial z\, \partial r} + r^{f_1(z)-1}\tilde{\Phi}_1^{**}(\theta, Tz, f_1(z)) + r^{f_1(z)-1}\frac{\partial f_1(z)}{\partial z}\ln r\Phi_1^{***}(\theta, Tz, f_1(z)). \tag{23}$$

When $Tz = 0$ or $Tz = 0.5$, there is no energy present in the z direction and Rice's J-integral is path independent. Therefore, the character of singularity of stresses given by HRR is right. In this case, the exponent of r in the stress function keeps greater than unity. When $0 < Tz < 0.5$, it will be shown in part II that the singularity of stresses is weaker than that for $Tz = 0$ or 0.5 because of the presence of energy in the z direction. In other words the exponent of r in stress functions will be larger. Consequently, in the range of $0 \leqslant Tz \leqslant 0.5$ the following inequality is tenable:

$$f_1(z) > 1.$$

Then there exist the following limits:

$$\lim_{r\to 0} r^{f_1(z)}\ln r = -\lim_{r\to 0}\frac{r^{f_1(z)}}{f_1(z)} \sim r^{f_1(z)}$$

$$\lim_{r\to 0} r^{f_1(z)-1}\ln r = -\lim_{r\to 0}\frac{r^{f_1(z)-1}}{f_1(z)-1} \sim r^{f_1(z)-1}.$$

With these limits, eq. (23) can be simplified near the crack tip as

$$r^{f_{yz}(z)}\tilde{\sigma}_{yz}(\theta, Tz) = r^{f_1(z)-1}\tilde{\Phi}_1^{**}(\theta, Tz) + 0(r^{f_1(z)}),$$

where the symbol $0(x)$ means that when $x \to 0$, $0(x) \to$constant or zero. Therefore

$$\Phi_1 = r^{f_{yz}(z)+1}\tilde{\Phi}_1(\theta, Tz). \tag{24}$$

Similarly, from eq. (16) it can be obtained that

$$\Phi_2 = r^{f_{xz}(z)+1}\tilde{\Phi}_2(\theta, Tz). \tag{25}$$

Substituting (18), (24) and (25) into the first three equations of (6) gives

$$\sigma_{xx} = r^{f_{xy}(z)}\tilde{\Phi}_{31}(\theta, Tz) + r^{f_{xz}(z)+1}\frac{\partial\tilde{\Phi}_2}{\partial z} + 0(r^{f_{xy}(z)+1}) + 0(r^{f_{xz}(z)+2})$$

$$\sigma_{yy} = r^{f_{xy}(z)}\tilde{\Phi}_{32}(\theta, Tz) + r^{f_{xz}(z)+1}\frac{\partial\tilde{\Phi}_2}{\partial z} + 0(r^{f_{xy}(z)+1}) + 0(r^{f_{xz}(z)+2})$$

$$\sigma_{zz} = r^{f_{xz}(z)-1}\tilde{\Phi}_{22}(\theta, Tz) + r^{f_{yz}(z)-1}\tilde{\Phi}_{12}(\theta, Tz). \tag{26}$$

According to hypothesis 1, the order of singularity of σ_{zz} is the same as σ_{xx} and σ_{yy}, so from eq. (26) it can be obtained that

$$f_{xy}(z) \leqslant f_{xz}(z) - 1$$

$$f_{xy}(z) \leqslant f_{yz}(z) - 1$$

and at least one of these equations should hold up. Therefore, without loss of generality it can be assumed that

$$f_{xy}(z) = f_{xz}(z) - 1 = f_{yz}(z) - 1 = f(z) - 2. \tag{27}$$

Substituting (27) into (18), (24) and (25) leads to

$$\Phi_i(\theta, r, z) = r^{f(x)}\tilde{\Phi}_i(\theta, Tz) \quad (i = 1, 2, 3). \tag{28}$$

Further, as $f(z) > 1$, and $\lim_{r\to 0} r^{f(z)-1}\ln r \sim r^{f(z)-1}$, so it can be obtained from eq. (28) that

$$\frac{\partial^2\Phi_1}{\partial z^2}, \quad \frac{\partial^2\Phi_2}{\partial z^2} \sim r^{f(z)-1} \sim 0(0)$$

$$\frac{\partial^2\Phi_3}{\partial x^2}, \quad \frac{\partial^2\Phi_3}{\partial y^2} \sim r^{f(z)-2}.$$

It follows that the singular terms of stresses σ_{xx} and σ_{yy} are only related to Φ_3. Therefore, in the asymptotic field at 3D crack tips, eq. (6) can be simplified as

$$\sigma_{xx} = \frac{\partial^2\Phi_3}{\partial y^2}$$

$$\sigma_{yy} = \frac{\partial^2\Phi_3}{\partial x^2}$$

$$\sigma_{zz} = Tz\left(\frac{\partial^2\Phi_3}{\partial x^2} + \frac{\partial^2\Phi_3}{\partial y^2}\right)$$

$$\sigma_{xy} = -\frac{\partial^2\Phi_3}{\partial x\,\partial y}$$

$$\sigma_{yz} = -\frac{\partial^2\Phi_1}{\partial y\,\partial z}$$

$$\sigma_{xz} = -\frac{\partial^2\Phi_2}{\partial x\,\partial z}. \tag{29}$$

Substituting (28) into (29), the asymptotic solution for stresses can be obtained

$$\sigma_{xx} = r^{f(z)-2}\tilde{\sigma}_{xx}(\theta, Tz)$$
$$\sigma_{yy} = r^{f(z)-2}\tilde{\sigma}_{yy}(\theta, Tz)$$
$$\sigma_{zz} = r^{f(z)-2}\tilde{\sigma}_{zz}(\theta, Tz) = Tzr^{f(z)-2}(\tilde{\sigma}_{xx} + \tilde{\sigma}_{yy})$$
$$\sigma_{xy} = r^{f(z)-2}\tilde{\sigma}_{xy}(\theta, Tz)$$
$$\sigma_{yz} = r^{f(z)-1}\tilde{\sigma}_{yz}(\theta, Tz) \sim 0(0)$$
$$\sigma_{xz} = r^{f(z)-1}\tilde{\sigma}_{xz}(\theta, Tz) \sim 0(0). \tag{30}$$

Obviously, the in-plane stresses $(\sigma_{xx}, \sigma_{yy}, \sigma_{xy})$ and σ_{zz} are singular with the order of singularity $f(z) - 2$, which is a function of $T(z)$. The stresses σ_{yz} and σ_{xz} are of the order of unity, and so can be ignored in the asymptotic analysis. Therefore the singular stresses can be determined by the stress function Φ_3 alone.

3.4. *The singularity of crack border strains*

(1) *Singularity of strains determined from strain–stress relations.* As $n > 1$, substituting (30) into the strain–stress relations in (7) can yield the dominant terms of strains

$$\varepsilon_{ij} = r^{n(f(z)-2)}\tilde{\varepsilon}_{ij}(\theta, Tz) \quad (i, j = x, y)$$
$$\varepsilon_{yz} = r^{n(f(z)-1)+1}\tilde{\varepsilon}_{yz}(\theta, Tz)$$
$$\varepsilon_{xz} = r^{n(f(z)-1)+1}\tilde{\varepsilon}_{xz}(\theta, Tz) \tag{31}$$

and

$$\varepsilon_{zz} = (Tz - \tfrac{1}{2})r^{n(f(z)-2)}\tilde{\varepsilon}_{zz}(\theta, Tz). \tag{32}$$

Evidently the in-plane strains ε_{xx}, ε_{yy} and ε_{xy} determined by strain–stress relations have singularity of the order of $n(f(z) - 2)$, while ε_{yz} and ε_{xz} have singularity one order lower, or $n(f(z) - 2) + 1$. This is similar to the plane strain problem. However, the singularity obtained here for 3D crack border strains keeps changing with $T(z)$, and the effect of $T(z)$ upon ε_{zz} is much stronger. At the corner points where the crack front penetrates the free surfaces, ε_{zz} has the same singularity as the in-plane strains, or of the order $n(f(z) - 2)$. So, in these regions, large out-of-plane deformation will occur. This has been proved by experiments [16, 20]. In the interior border of the crack, the singularity of ε_{zz} will be at least two orders lower when the triaxial stress constraint becomes strongest, or $Tz = \frac{1}{2} - br^2$. Consequently, it is difficult to determine ε_{zz} accurately by the asymptotic field itself because its singularity may have a change of two orders along the crack front, and (32) is a first approximation.

(2) *Requirement of the compatibility of strains.* To investigate the correctness of the above analyses it is necessary to check whether the compatibility equations (5) are satisfied. Substituting (31) and (32) into (5), it is found that all equations in eq. (5) for $ki \neq zz$ can be satisfied automatically by the structure of strains presented by (31). The others associated with ε_{zz} have further requirements on ε_{zz} that

$$\frac{\partial^2 \varepsilon_{zz}}{\partial r^2} \sim (\ln r)^2 r^{n(f(z)-2)}\varphi_1 + \ln r r^{n(f(z)-2)}\varphi_2, \tag{33}$$

where $\varphi_i (i = 1, 2)$ are functions of θ, Tz and differentiations of Tz.

As

$$-1 < -\frac{n}{n+1} \leqslant n(f(z) - 2) < 0,$$

it can be obtained by integrating both sides of (33) that the only singular term of ε_{zz} is

$$\varepsilon_{zz} \sim (\ln r)^2 r^{n(f(z)-2)+2}\varphi(\theta, Tz). \tag{34}$$

This singularity is weaker than $r^{n(f(z)-2)}$ and stronger than $r^{n(f(z)-2)+2}$, or it is between the upper and lower limits of (32). So (32) is a good first approximation of the problem.

Summarizing the above structures of crack border stresses and strains, it can be found that the solutions have similar forms to the planar solution except for the influence of Tz. Thus in view of the mathematical solving process, the problem may be simplified to a quasi-planar problem with the triaxial stress constraint Tz being considered providing the special boundary conditions and symmetry requirements of the 3D problem can also be satisfied.

3.5. *Boundary conditions and energy requirements*

(1) *Boundary conditions and symmetry requirement.* To investigate whether a 3D cracked problem can be simplified to a quasi-planar problem in the x–y plane, assume σ_{ij} $(i, j = x, y)$ satisfying the boundary conditions and symmetry requirement of the corresponding 2D problem, or assume that

on the free crack surfaces $(y = 0, x < 0)$

$$\sigma_{yy} = 0, \quad \sigma_{xy} = 0 \tag{35}$$

on the x–z plane $(y = 0, x > 0)$

$$\frac{\partial \sigma_{yy}}{\partial y} = \frac{\partial \sigma_{xx}}{\partial y} = 0; \quad \sigma_{xy} = 0. \tag{36}$$

For a through-thickness mode I crack, Tz is of the following character as mentioned in Section 2:

$$Tz = 0 \quad \left(z = \pm\frac{B}{2}\right) \tag{37}$$

$$\frac{\partial Tz}{\partial z} = 0 \quad (z = 0) \tag{38}$$

$$\frac{\partial Tz}{\partial y} = 0 \quad (y = 0). \tag{39}$$

Then let us investigate the satisfaction of other boundary conditions and symmetry requirements of the 3D problem.

On the symmetry plane $(z = 0)$, it is required that

$$\frac{\partial \sigma_{zz}}{\partial z} = 0, \quad \sigma_{yz} = \sigma_{xz} = 0. \tag{40}$$

As

$$\frac{\partial \sigma_{ii}}{\partial z} = r^{f(z)-2} \ln r \frac{\partial f(z)}{\partial Tz} \frac{\partial Tz}{\partial z} \tilde{\sigma}_{ii}(\theta, Tz) + r^{f(z)-2} \frac{\partial \tilde{\sigma}_{ii}}{\partial Tz} \frac{\partial Tz}{\partial z} \quad (ii = xx, yy, zz),$$

so it can be shown from (38) that

$$\frac{\partial \sigma_{zz}}{\partial z} = 0 \quad \text{for } z = 0.$$

Similarly, the tenability of other requirements in (40) can be proved using (30), (29) and (38). In fact, as σ_{yz} and σ_{xz} have the order of unity, these requirements need not be considered.

On the x–z plane $(y = 0, x > 0)$, it is required that

$$\frac{\partial \sigma_{zz}}{\partial y} = 0, \quad \frac{\partial \sigma_{yz}}{\partial y} = 0. \tag{41}$$

From (36) and (39), it can be obtained for $y = 0$ and $x > 0$ that

$$\frac{\partial \sigma_{zz}}{\partial y} = Tz\left(\frac{\partial \sigma_{xx}}{\partial y} + \frac{\partial \sigma_{yy}}{\partial y}\right) + \frac{\partial Tz}{\partial y}(\sigma_{xx} + \sigma_{yy}) = 0.$$

As $\lim_{r \to 0} \sigma_{yz} \equiv 0$, the last equation in (41) is also tenable.

On the free surfaces $(z = \pm B/2)$, it is required that

$$\sigma_{zz} = 0, \quad \sigma_{xz} = \sigma_{yz} = 0. \tag{42}$$

It is known from (37) that

$$\sigma_{zz} = Tz(\sigma_{xx} + \sigma_{yy}) = 0 \quad \left(z = \pm\frac{B}{2}\right).$$

The last two requirements can also be satisfied near the crack tip as σ_{yz} and σ_{xz} have the order of unity.

The above boundary conditions are for a through thickness crack. In fact, for general 3D mode I cracks with the crack border normal to the free surfaces, the satisfaction of boundary conditions and symmetry requirements can also be proved in the same way. Therefore, a 3D crack problem can be simplified as a quasi-planar problem in the x–y plane with the triaxial stress constraint Tz being considered. As ε_{yz} and ε_{xz} have singularity one order weaker than that of the in-plane strains, and all of the singular stresses are determined by one stress function Φ_3, so the main compatibility of the quasi-planar problem is

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = \frac{\partial^2 \varepsilon_{xy}}{\partial x\, \partial y}. \tag{43}$$

(2) *Requirements of finiteness of energy.* When there exist singular points, solutions satisfying all of the differential equations and boundary conditions are not unique. A true solution should also satisfy the requirement of finiteness of energy W at the singular points. For 3D cracks, the requirement can be expressed as

$$W = \iiint \frac{\mathrm{d}W}{\mathrm{d}V} r\, \mathrm{d}r\, \mathrm{d}\theta\, \mathrm{d}z, \quad r \to 0 \tag{44}$$

is bounded, where $\mathrm{d}W/\mathrm{d}V$ is the strain energy density

$$\frac{\mathrm{d}W}{\mathrm{d}V} = \int_0^{\varepsilon_{ij}} \sigma_{ij}\, \mathrm{d}\varepsilon_{ij}.$$

Substituting (30)–(32) into (44), it can be obtained that

$$W = \iiint \left[\int_0^{\varepsilon_{ij}} r^{f(z)-2} \tilde{\sigma}_{ij}\, \mathrm{d}(r^{n(f(z)-2)} \tilde{\varepsilon}_{ij})\right] r\, \mathrm{d}r\, \mathrm{d}\theta\, \mathrm{d}z. \tag{45}$$

Assume $f(z)$ keeps unchanged in the loading process; then

$$W = \iiint \frac{F(\theta, Tz)}{r^{[2-f(z)](n+1)-1}} \mathrm{d}r\, \mathrm{d}\theta\, \mathrm{d}z, \quad r \to 0. \tag{46}$$

According to singular integral theory, the boundedness of (46) can be guaranteed only if $[2 - f(z)](n+1) - 1 < 1$, or

$$f(z) > \frac{2n}{n+1}. \tag{47}$$

In the HRR solution, $Tz = 0$ or 0.5,

$$f(z) \equiv \frac{2n+1}{n+1} > \frac{2n}{n+1}.$$

When $0 < Tz < \frac{1}{2}$, it will be shown in part II that $f(z) > (2n+1)/(n+1)$, so (47) is tenable.

Actually, $f(z)$ may change with loading process, and (45) cannot be expressed as (46). But for $f(z)$ never less than $(2n+1)/(n+1)$, according to the mean value theorem of integration, the boundedness of (45) can still be guaranteed.

4. DEGENERATION OF THE SOLUTION INTO PLANE PROBLEMS

As a special case of 3D problem, the planar crack tip fields should be able to be obtained from the above solution under stricter confinement.

4.1. Structures of crack border stresses

In Section 3, the singular structure of Φ_1 is determined by the use of limit analyses following eq. (23). If the confinement is enforced and letting

$$\frac{\partial f_1(z)}{\partial z} = 0, \tag{48}$$

both sides of (23) can also be ensured to have the same form, and so (48) is a sufficient requirement.

When (48) is used, the dominant term of stress functions can be obtained in a similar manner as in Section 3 that

$$\Phi_i(\theta, r, z) = r^k \tilde{\Phi}_i(\theta, Tz) \quad (i = 1, 2, 3). \tag{49}$$

From (49) it is easy to obtain the singular stresses

$$\begin{aligned}
\sigma_{xx} &= \frac{\partial^2 \Phi_3}{\partial y^2} = r^{k-2}\tilde{\sigma}_{xx}(\theta, Tz) \\
\sigma_{yy} &= \frac{\partial^2 \Phi_3}{\partial x^2} = r^{k-2}\tilde{\sigma}_{yy}(\theta, Tz) \\
\sigma_{zz} &= Tz\left(\frac{\partial^2 \Phi_3}{\partial x^2} + \frac{\partial^2 \Phi_3}{\partial y^2}\right) = r^{k-2}\tilde{\sigma}_{zz}(\theta, Tz) \\
\sigma_{xy} &= -\frac{\partial^2 \Phi_3}{\partial x\, \partial y} = r^{k-2}\tilde{\sigma}_{xy}(\theta, Tz) \\
\sigma_{yz} &= -\frac{\partial^2 \Phi_1}{\partial y\, \partial z} = r^{k-1}\tilde{\sigma}_{yz}(\theta, Tz) \sim 0(0) \\
\sigma_{xz} &= -\frac{\partial^2 \Phi_2}{\partial x\, \partial z} = r^{k-1}\tilde{\sigma}_{xz}(\theta, Tz) \sim 0(0),
\end{aligned} \tag{50}$$

where k is constant and the singularities of stresses are independent of Tz.

4.2. The singularity of crack border strains

Substituting (50) into (7) leads to

$$\begin{aligned}
\varepsilon_{ij} &= r^{n(k-2)}\tilde{\varepsilon}_{ij}(\theta, Tz) \quad (i, j = x, y) \\
\varepsilon_{yz} &= r^{n(k-2)+1}\tilde{\varepsilon}_{yz}(\theta, Tz) \\
\varepsilon_{xz} &= r^{n(k-2)+1}\tilde{\varepsilon}_{xz}(\theta, Tz)
\end{aligned} \tag{51}$$

and

$$\varepsilon_{zz} = (Tz - \tfrac{1}{2}) r^{n(k-2)}\tilde{\varepsilon}_{zz}(\theta, Tz). \tag{52}$$

Substituting (51) and (52) into (5), it can be found that to satisfy all of the compatibility equations, Tz must satisfy the following attached requirement:

$$Tz = \tfrac{1}{2} - br^2, \tag{53}$$

where b is a parameter there cannot be determined by the asymptotic field.

Substituting (53) into (52) yields

$$\varepsilon_{zz} \sim br^{n(k-2)+2}, \tag{54}$$

that is, $\varepsilon_{zz} = 0$ when $b = 0$, or ε_{zz} will be two orders weaker than in-plane strains $(\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{xy})$.

4.3. Discussion

As Tz and all of the components of stresses and strains remain unchanged with z, the above solutions are those of plane problems.

(1) *General plane stress solution*. Let $\sigma_{zz} = 0$, $Tz = 0$, then all of the basic differential equations listed in Section 3.1 can be satisfied except the three compatibilities associated with ε_{zz}. In such a case, ε_{zz} has the same singularity as the in-plane strains, or $r^{n(k-2)}$, while shear strains ε_{xz} and ε_{yz}

and shear stresses σ_{xz} and σ_{yz} are one order weaker than their in-plane correspondents, and so can be neglected. Therefore, the problem has degenerated to a general plane stress problem. This indicates that the plane stress HRR solution is the exact solution of the general plane stress problem.

(2) *Exact solution of plane strain cracks.* When the attached requirement (53) is satisfied, all of the basic differential equations can be satisfied. Then σ_{zz} has the same singularity as in-plane stresses. And σ_{xz} and σ_{yz} are one order weaker, and so can be ignored. The strain components ε_{xz}, ε_{yz} and ε_{zz} are also neglectable compared with their in-plane correspondents. Consequently, (50)–(53) give the exact asymptotic solution for plane strain cracks.

Equation (53) gives the general expression of the triaxial stress constraint at plane strain crack tips, where b is dependent upon the full environment of loading. As Tz must be equal to Poisson's ratio ν of the material in the elastic region for plane strain, so the global solution about the boundary of the plastic zone $r_p(\theta)$ may be helpful to determine b.

If (53) holds good in the plastic region, it can be obtained that

$$b = \frac{\frac{1}{2} - \nu}{r_p^2(\theta)}.$$

So, formula (53) becomes

$$Tz = \tfrac{1}{2} - (\tfrac{1}{2} - \nu)\left(\frac{r}{r_p(\theta)}\right)^2. \tag{55}$$

As $\nu < \frac{1}{2}$, Tz is a decreasing function of r, but it increases with r_p. When $r \to 0$, Tz reaches its upper limit of $\frac{1}{2}$. This tendency of Tz is coincident with the concept of elastic–plastic mechanics, in which the elastic–plastic Poisson ratio ν_{ep} that is equivalent to Tz in plane strain reaches its upper limit, 0.5, when the ratio of the plastic strain to elastic strain $\varepsilon^p/\varepsilon^e \to \infty$, which is the situation as $r \to 0$ in the singular crack tip field, and Tz decreases to ν when the plastic strain dies out.

Furthermore, it can be seen that HRR solution for $Tz = 0.5$ is an exact solution for plane strain, but not a real solution for actual specimens. As $Tz = 0.5$ gives the upper limit of the stress triaxiality constraint, the HRR field will be a limit distribution as well. The difference of the HRR field from that of real specimens will change with the distribution of $r_p(\theta)$, which in turn is dependent upon the global solution. This gives a theoretical explanation for the dependence of HRR dominance on load levels and the type of specimens.

Therefore, eq. (53) has important physical significance that the asymptotic field at a plane strain crack tip is not independent of, but closely connected to, the global environment. A single parameter cannot characterize the plane strain crack tip field very well.

In plane stress, $Tz \equiv 0$, this connection is cut out and a relatively independent asymptotic field exists.

5. CONCLUSIONS

(1) The triaxial stress constraint Tz is the essential reason for the defects of 2D solutions in the description of 3D cracks.

(2) In the present paper, the structure of predominant singular stresses and strains has been obtained as follows:

$$\sigma_{ij} = r^{f(z)-2}\tilde{\sigma}_{ij}(\theta, Tz) \quad (i, j = x, y)$$

$$\sigma_{zz} = Tz(r, \theta, z)(\sigma_{xx} + \sigma_{yy})$$

$$\sigma_{xz},\ \sigma_{yz} \sim 0(0)$$

and

$$\varepsilon_{ij} = r^{n(f(z)-2)}\tilde{\varepsilon}_{ij}(\theta, Tz) \quad (i, j = x, y)$$

$$\varepsilon_{xz},\ \varepsilon_{yz} \sim 0(r^{n(f(z)-2)+1}).$$

ε_{zz} has the same singularity as ε_{ij} $(i, j = x, y)$ at the corner points, while in the interior of the crack border its singularity may become very much weaker, even disappear.

(3) It is proved that the problem can be simplified to a quasi-planar problem with the triaxial stress constraint T_z being taken into account. This makes it possible to solve the problem analytically.

(4) When T_z is present, the local field at the crack tips will no longer be independent of the far field. Thus to characterize the local field, T_z must be introduced to take account of the effect of global environments of stress configurations.

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